CCNY-HEP-96/4

February 1996

W_∞ AND w_∞ GAUGE THEORIES AND CONTRACTION

An. Kavalov and B. Sakita*

Physics Department, City College of the City University of New York

New York, NY 10031

Abstract

We present a general method of constructing W_∞ and w_∞ gauge theories in terms of $d+2$ dimensional local fields. In this formulation the W_∞ gauge theory Lagrangians involve non-local interactions, but the w_∞ theories are entirely local. We discuss the so-called classical contraction procedure by which we derive the Lagrangian of w_∞ gauge theory from that of the corresponding W_∞ gauge theory. In order to discuss the relationship between quantum W_∞ and quantum w_∞ gauge theory we solve $d = 1$ gauge theory models of a Higgs field exactly by using the collective field method. Based on this we conclude that the W_∞ gauge theory can be regarded as the large N limit of the corresponding $SU(N)$ gauge theory once an appropriate coupling constant renormalization is made, while the w_∞ gauge theory cannot be.

*e-mail address: kavalov@scisun.sci.ccny.cuny.edu; sakita@scisun.sci.ccny.cuny.edu

1. Introduction and Summary

W_∞ algebra and its so-called classically contracted w_∞ algebra appeared recently in various problems in physics, in particular in the study of $c = 1$ string theory [1] and quantum Hall system [2, 3]. The gauge theory based on these algebras also appeared in these studies [4, 5]. In fact the same algebras and the gauge theories based on them had been proposed previously as the theories [6, 7, 8] relevant to the large N limit of $SU(N)$ gauge theories [9]. But it seems to us that not enough studies have been done to differentiate the formal and dynamic aspect of W_∞ and w_∞ theories. In view of the recent developments we study in this paper this subject as systematically as possible by using the technique developed in the study of quantum Hall system [2, 5].

The W_∞ algebra is a commutator algebra of Hermitian operators of one harmonic oscillator [10]. It is an infinite-dimensional Lie algebra. If we choose a set of linearly independent real function of z and $\bar{z}$ as the parameters of W_∞ group, the structure constants of the algebra are expressed in terms of Moyal bracket [11]. Replacing the Moyal bracket by a Poisson bracket we define the w_∞ algebra. It is an algebra of area-preserving diffeomorphism. As an introduction we discuss this issue in section 2 together with the so-called classical contraction procedure by which the W_∞ algebra is transformed to the w_∞ algebra.

W_∞ gauge theory is a gauge field theory of W_∞ as an internal symmetry algebra. The W_∞ gauge potential is a space-time dependent Hermitian operator of harmonic oscillator. In the coherent state representation it is a function of z, $\bar{z}$, which we call the color space coordinates, and x^μ $(\mu = 1, 2, \ldots d)$, the space time coordinates. Thus, we can express the W_∞ gauge theories in terms of $d + 2$ dimensional local fields. The interactions of the fields are necessarily non-local in the color space in W_∞ theories, but they are local in w_∞ theories. In section 3 we define W_∞ theories as $d + 2$ dimensional field theories with non-local interactions and w_∞ gauge theories as $d + 2$ dimensional local field theories. Since the W_∞ algebra is closely related to the w_∞ algebra, the gauge theories based on

these algebras may also be closely related. In order to see the relationship at classical Lagrangian level, we introduce the $l \to 0$ contraction procedure by which we derive the w_∞ gauge theories from the corresponding W_∞ gauge theories. The procedure consists of the introduction of a length scale l in the color space, an appropriate scale transformation of the fields, and the $l \to 0$ limit. In this section we also introduce matter fields analogous to the quark fields and the Higgs fields.

The W_∞ algebra can be considered as the $N \to \infty$ limit of the $SU(N)$ algebra [10]. Therefore, the W_∞ gauge theory or its variation w_∞ gauge theory [7, 12] might be used for the large N gauge theory. Since the w_∞ gauge theory is a local theory and much easier to be handled, it is important to determine whether this theory can serve for the large N gauge theory or not. For this purpose we solve $d = 1$ gauge theory of Higgs field exactly in section 4, which reveals also the quantum mechanical relationship between the W_∞ theory and the w_∞ theory. In $d = 1$ there exists only the time component of the gauge potential and the pure gauge theory is trivial but it constrains the states of Higgs field to be gauge invariant. The W_∞ theory becomes essentially $N = \infty$ limit of $d = 1$ gauged matrix model. On the other hand the w_∞ model becomes an infinite number of non-interacting quantum mechanical systems. We use the collective field method [13] to solve these theories. The spectrum of these two theories are in general different and coincides with the $N = \infty$ limit of $SU(N)$ theory only for W_∞ model. Based on this result we conclude that W_∞ gauge theory can but w_∞ gauge theory cannot serve for the purpose of large N gauge theory.

2. W_∞ and w_∞ Algebra.

We define the W_∞ algebra as a commutator algebra of all Hermitian operators $\xi(\hat{a}, \hat{a}^\dagger)$ in the Hilbert space of a harmonic oscillator [2]. A convenient parametrization for these operators is achieved by using a real function $\xi(z, \bar{z})$ as

$$\xi(\hat{a}, \hat{a}^\dagger) = \ddagger \xi(z, \bar{z})\Big|_{\substack{z=\hat{a} \\ \bar{z}=\hat{a}^\dagger}} \ddagger = \int d^2 z e^{-|z|^2} |z\rangle \xi(z, \bar{z}) \langle z|, \tag{2.1}$$

where $\hat{a}^\dagger$ and $\hat{a}$ are standard creation and annihilation operators and $\ddagger \quad \ddagger$ stands for

the anti-normal-order symbol, i.e., all the creation operators stand to the right of the annihilation operators. The last expression (2.1) is in the coherent state representation (see A1). Obviously the product of two ξ's is not anti-normally ordered and we bring it to the anti-normal-order form by using the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. We obtain (see A.1)

$$\xi_1(\hat{a}, \hat{a}^\dagger)\xi_2(\hat{a}, \hat{a}^\dagger) = \ddagger \sum_{n=1}^{\infty} \frac{(-)^n}{n!} \partial_{\bar{z}}^n \xi_1(z, \bar{z}) \partial_z^n \xi_2(z, \bar{z}) \Big|_{\substack{z=\hat{a} \\ \bar{z}=\hat{a}^\dagger}} \ddagger, \tag{2.2}$$

from which the following commutation relation follows

$$[\xi_1(\hat{a}, \hat{a}^\dagger), \xi_2(\hat{a}, \hat{a}^\dagger)] = i\{\!\{\xi_1, \xi_2\}\!\}(\hat{a}, \hat{a}^\dagger), \tag{2.3}$$

where $\{\!\{\xi_1, \xi_2\}\!\}$ is a Moyal bracket [11] defined by

$$\{\!\{\xi_1, \xi_2\}\!\}(z, \bar{z}) \equiv i \sum_{n=1}^{\infty} \frac{(-)^n}{n!} \left(\partial_z^n \xi_1(z, \bar{z}) \partial_{\bar{z}}^n \xi_2(z, \bar{z}) - \partial_{\bar{z}}^n \xi_1(z, \bar{z}) \partial_z^n \xi_2(z, \bar{z}) \right). \tag{2.4}$$

The commutation relation (2.3) is that of the W_∞ Lie algebra in the fundamental representation. The W_∞ is an infinite-dimensional Lie group with parameters being a set of linearly independent real functions $\xi(z, \bar{z})$. The generators of W_∞ are the linear functionals of $\xi(z, \bar{z})$. Thus we write for arbitrary representation:

$$[\rho[\xi_1], \rho[\xi_2]] = i\rho[\{\!\{\xi_1, \xi_2\}\!\}], \tag{2.5}$$

where ρ is the generator of W_∞ group.

The Lie algebra of w_∞ , the area-preserving diffeomorphisms, is defined by the commutation relation

$$[\rho[\xi_1], \rho[\xi_2]] = i\rho[\{\xi_1, \xi_2\}], \tag{2.6}$$

where $\{\ ,\ \}$ is the Poisson bracket symbol.

It is well known [2] that one can obtain the w_∞ algebra from the W_∞ by a contraction. To explain it let us introduce a length scale l in the $z, \bar{z}$ space, which we call the color space, and set

$$z = \frac{1}{\sqrt{2}l}(\sigma_x + i\sigma_y), \qquad \bar{z} = \frac{1}{\sqrt{2}l}(\sigma_x - i\sigma_y). \tag{2.7}$$

The Poisson bracket is the leading surviving term of the Moyal bracket in the $l \to 0$ limit. To be more specific we set $\xi(z,\bar{z}) = l^{-2}\xi(\vec{\sigma})$ and obtain

$$\begin{aligned}\lim_{l\to 0} l^2 \{\!\{\xi_1,\xi_2\}\!\}(z,\bar{z}) &= \partial_{\sigma_x}\xi_1(\vec{\sigma})\partial_{\sigma_y}\xi_2(\vec{\sigma}) - \partial_{\sigma_y}\xi_1(\vec{\sigma})\partial_{\sigma_x}\xi_2(\vec{\sigma}) \\ &\equiv \epsilon^{ij}\partial_i\xi_1(\vec{\sigma})\partial_j\xi_2(\vec{\sigma}) \\ &\equiv \{\xi_1,\xi_2\}(\vec{\sigma}).\end{aligned} \tag{2.8}$$

In this paper we call this procedure as the $l \to 0$ contraction.

3. W_∞ and w_∞ Gauge Invariant Lagrangians

The W_∞ gauge theory is a gauge field theory of W_∞ as an internal symmetry algebra.

Let us discuss first the pure Yang-Mills theory. We introduce a gauge potential $\hat{\mathcal{A}}_\mu$ which is a Hermitian operator in the harmonic oscillator Hilbert space as well as a function of space time:

$$\hat{\mathcal{A}}_\mu(x) \equiv \mathcal{A}_\mu(x,\hat{a},\hat{a}^\dagger) = \int d^2z e^{-|z|^2}|z\rangle\mathcal{A}_\mu(x,z,\bar{z})\langle z|. \tag{3.1}$$

The action is given by

$$S_{YM} = -\frac{1}{4g^2}\int d^dx \ \mathrm{tr}\big(\hat{\mathcal{F}}_{\mu\nu}\hat{\mathcal{F}}^{\mu\nu}\big), \tag{3.2}$$

where $\hat{\mathcal{F}}_{\mu\nu}$ is the field strength defined by

$$\hat{\mathcal{F}}_{\mu\nu} = \partial_\mu\hat{\mathcal{A}}_\nu - \partial_\nu\hat{\mathcal{A}}_\mu - i[\hat{\mathcal{A}}_\mu,\hat{\mathcal{A}}_\nu]. \tag{3.3}$$

We then rewrite these by using the coherent state representation:

$$S_{YM} = -\frac{1}{4g^2}\int\!\!\int d^dx d^2z \sum_{n=0}^{\infty}\frac{(-)^n}{n!}\partial_z^n\mathcal{F}_{\mu\nu}(x,z,\bar{z})\partial_{\bar{z}}^n\mathcal{F}^{\mu\nu}(x,z,\bar{z}), \tag{3.4}$$

where

$$\mathcal{F}_{\mu\nu}(x,z,\bar{z}) = \partial_\mu\mathcal{A}_\nu(x,z,\bar{z}) - \partial_\nu\mathcal{A}_\mu(x,z,\bar{z}) + \{\!\{\mathcal{A}_\mu,\mathcal{A}_\nu\}\!\}(x,z,\bar{z}). \tag{3.5}$$

This action is invariant under the W_∞ gauge transformation:

$$\delta\hat{\mathcal{A}}_\mu(x) = \partial_\mu\hat{\xi}(x) + i[\hat{\xi}(x),\hat{\mathcal{A}}_\mu(x)], \qquad \delta\mathcal{A}_\mu(x,z,\bar{z}) = \partial_\mu\xi(x,z,\bar{z}) - \{\!\{\xi,\mathcal{A}_\mu\}\!\}(x,z,\bar{z}). \tag{3.6}$$

In the coherent sate representation the gauge fields are formally $d+2$ dimensional local fields, d for space time and 2 for color space. However, the interactions are non-local since the action involves derivatives of infinite order.

In the action (3.4) we no longer have damping factor $\mathrm{e}^{-|z|^2}$ because it is a trace expression. Therefore we have to restrict the field configurations so that we can integrate by parts in the color space. In this paper we define our W_∞ gauge theories as $d+2$ dimensional field theories such that all the fields and their derivatives vanish at $z=\infty$.

Next, let us introduce a fermion field, which is a fundamental representation of W_∞ , namely a field which transforms as a bra or ket vector in the Hilbert space of harmonic oscillator:

$$|\psi(x)\rangle = \int |z\rangle d^2 z e^{-|z|^2} \langle z|\psi(x)\rangle \equiv \int |z\rangle\ d^2 z e^{-|z|^2} \psi(x,\bar{z}). \tag{3.7}$$

We write the action as

$$\begin{aligned} S_F &= \int d^n x \langle\psi(x)|\ \gamma^\mu \big(i\partial_\mu - \hat{\mathcal{A}}_\mu(x)\big)|\psi(x)\rangle \\ &= \int\int d^n x d^2 z e^{-|z|^2} \bar{\psi}(x,z)\gamma^\mu\big(i\partial_\mu - \mathcal{A}_\mu(x,z,\bar{z})\big)\psi(x,\bar{z}), \end{aligned} \tag{3.8}$$

which is invariant under the W_∞ gauge transformation (3.6) and

$$\delta|\psi(x)\rangle = -i\hat{\xi}(x)|\psi(x)\rangle, \qquad\qquad \delta\psi(x,\bar{z}) = -i\ddagger\xi(\partial_{\bar{z}},\bar{z})\ddagger\psi(x,\bar{z}), \tag{3.9}$$

where $\ddagger\quad\ddagger$ indicates that the derivatives are placed on the left of $\bar{z}$.

As a last example of W_∞ gauge theory let us consider next a scalar field which is in an adjoint representation. In this paper we call this field simply a Higgs field.

$$\hat{M}(x) \equiv M(x,\hat{a},\hat{a}^\dagger) = \int d^2 z e^{-|z|^2} |z\rangle M(x,z,\bar{z})\langle z|. \tag{3.10}$$

The action is given by:

$$
\begin{aligned}
S_H &= \int d^d x \mathrm{tr}\left[\frac{1}{2}\big(\partial_\mu \hat{M}(x) - [\hat{\mathcal{A}}_\mu, \hat{M}](x)\big)\big(\partial^\mu \hat{M}(x) - [\hat{\mathcal{A}}^\mu, \hat{M}](x)\big) - v(\hat{M})\right] \\
&= \int d^d x \left[\int d^2 z \frac{1}{2}\sum_{m=0}^{\infty}\frac{(-)^m}{m!}\partial_z^m\big(\partial_\mu M(x,z,\bar{z}) - \{\!\{\mathcal{A}_\mu, M\}\!\}(x,z,\bar{z})\big)\times \right. \\
&\qquad\qquad \left. \times\partial_{\bar{z}}^m\big(\partial^\mu M(x,z,\bar{z}) - \{\!\{\mathcal{A}^\mu, M\}\!\}(x,z,\bar{z})\big)\right] - \mathrm{tr}v(\hat{M}), \\
v(\hat{M}) &= \sum_n g_n \hat{M}^n, \qquad\qquad dim(g_n) = d\left(\frac{n}{2} - 1\right) - n.
\end{aligned}
\tag{3.11}
$$

Here again we require that the fields and their $z, \bar{z}$ derivatives should fall off to zero at $z = \infty$. We can check that this action is invariant under the W_∞ gauge transformation (3.6) and

$$
\delta M(x,z,\bar{z}) = \{\!\{\xi, M\}\!\}(x,z,\bar{z}), \qquad\qquad \delta\hat{M}(x) = -i[\hat{\xi}(x), \hat{M}(x)]. \tag{3.12}
$$

Notice that here again the interactions are non-local in the color space.

The quantization of the theory is done by the standard canonical quantization. Although the interactions are non-local in the color space, they are local in the ordinary space. Accordingly there is no problem for the quantization. We shall see it explicitly in an example in the next section.

Let us discuss next the contraction procedure which will allow us to obtain the w_∞ gauge invariant actions from W_∞ ones. For the fields in the adjoint representation such as $\mathcal{A}_\mu$ and M this procedure is straightforward. As we mentioned earlier, by introducing two real coordinates σ_x and σ_y as in (2.6) and by taking the $l \to 0$ limit we reduce Moyal bracket to Poisson bracket. If we simultaneously rescale the fields and the coupling constants as

$$
\begin{aligned}
&\mathcal{A}_\mu(x,z,\bar{z}) = l^{-2}\mathcal{A}_\mu(x,\vec{\sigma}) \\
&M(x,z,\bar{z}) = (2\pi)^{\frac{1}{2}} l M(x,\vec{\sigma}) \\
&g^2 = \tilde{g}^2 l^{-6}, \qquad g_n = \tilde{g}_n (2\pi)^{\frac{1}{2}} l^{2-n},
\end{aligned}
\tag{3.13}
$$

we obtain from (3.4) and (3.11) the following w_∞ gauge invariant $d+2$ dimensional local

field theory:

$$
\begin{aligned}
S_{YM} &= -\frac{1}{4\tilde{g}^2}\int d^dx d^2\vec{\sigma} F_{\mu\nu}(x,\vec{\sigma})F^{\mu\nu}(x,\vec{\sigma}),\\
&\qquad F_{\mu\nu}(x,\vec{\sigma}) = \partial_\mu \mathcal{A}_\nu(x,\vec{\sigma}) - \partial_\nu \mathcal{A}_\mu(x,\vec{\sigma}) + \epsilon^{ij}\partial_i \mathcal{A}_\mu(x,\vec{\sigma})\partial_j \mathcal{A}_\nu(x,\vec{\sigma});\\
S_H &= \int d^dx d^2\vec{\sigma}\left[\frac{1}{2}\big(\partial_\mu M(x,\vec{\sigma}) - \epsilon^{ij}\partial_i \mathcal{A}_\mu(x,\vec{\sigma})\partial_j M(x,\vec{\sigma})\big)\times\right.\\
&\qquad\qquad \left.\times\big(\partial^\mu M(x,\vec{\sigma}) - \epsilon^{ij}\partial_i \mathcal{A}^\mu(x,\vec{\sigma})\partial_j M(x,\vec{\sigma})\big) - \tilde{v}(M)\right],\\
&\qquad \tilde{v}(M) = \sum_n \tilde{g}_n M^n(x,\vec{\sigma}).
\end{aligned}
\tag{3.14}
$$

Here again we require that the fields vanish at $\vec{\sigma}=\infty$.

Setting $\xi(x,z,\bar{z}) = l^{-2}\xi(x,\vec{\sigma})$ we obtain the w_∞ gauge transformation:

$$
\begin{aligned}
\delta\mathcal{A}^\mu(x,\vec{\sigma}) &= \partial^\mu\xi(x,\vec{\sigma}) - \epsilon^{ij}\partial_i\xi(x,\vec{\sigma})\partial_j\mathcal{A}^\mu(x,\vec{\sigma})\\
\delta M(x,\vec{\sigma}) &= \epsilon^{ij}\partial_i\xi(x,\vec{\sigma})\partial_j M(x,\vec{\sigma}).
\end{aligned}
\tag{3.15}
$$

Even though the transformations (3.15) are obtained from (3.6) and (3.12) by the $l\to 0$ limit, we can independently check that the actions (3.14) is really invariant by the w_∞ gauge transformation (3.15). We remark that the second equation of (3.15) can be written as

$$
\delta M(x,\vec{\sigma}) = M(x,\vec{\sigma}+\delta\vec{\sigma}(x,\vec{\sigma})) - M(x,\vec{\sigma}), \qquad \delta\sigma^i(x,\vec{\sigma}) = -\epsilon^{ij}\partial_j\xi(x,\vec{\sigma}), \tag{3.16}
$$

which is a local area-preserving coordinate transformation.

As we mentioned earlier the damping factor $e^{-|z|^2}$ cancels out in Lagrangians for the fields in the adjoint representation such as $\mathcal{A}_\mu$ and M, due to the property of the trace in coherent state representation. But it remains there for the fields in fundamental representation, such as Fermi field (see (3.8)). Therefore, a naive $l\to 0$ limit leads to the trivial result ($S_F \equiv 0$). This may imply difficulty in introducing a Fermi field of fundamental representation in w_∞ theory.

We should mention that the YM lagrangian (3.14) had already been written down in the literature [7, 8].

4. One Dimensional Higgs Model: One Dimensional W_∞ and w_∞ Matrix model

In the previous section we presented a general method for constructing W_∞ gauge theory and then we obtained the w_∞ gauge theories by using the $l \to 0$ contraction procedure from W_∞ theories.

Several questions arise. Since W_∞ group can be considered as an $N = \infty$ limit of $SU(N)$ group, can one use the W_∞ gauge theory for the large N limit of $SU(N)$ gauge theory, especially for the large N QCD [9]? In the large N QCD one takes the $N \to \infty$ limit keeping $g^2 N$ finite. Since in W_∞ theories N is already at infinity, how can one implement the large N QCD condition ? A simple Feynman diagramatic calculation of w_∞ theory shows that the coupling constants are multiplicatively renormalized to absorb the infinite volume of color space. In a similar way in W_∞ guage theory the question arises whether or not one can implement the QCD condition as a multiplicative renormalization of coupling constants ? As shown in the previous section it is possible to obtain w_∞ theory from W_∞ theory by contraction. This shows the relationship between these theories as classical theories. How about in quantum theory ? Is there any physical region where one can use the w_∞ gauge theory for large N QCD ? In this section we address these questions by solving the simplest one-dimensional case exactly.

Since in $d = 1$ there exists only time component of gauge field and the pure gauge field model becomes trivial, we consider the gauge invariant Higgs model. This model may be thought of as a gauged one dimensional W_∞ (or w_∞) matrix model [14]. We solve it by using the collective field method [13]. Since we can carry out the discussions entirely in parallel for both W_∞ and w_∞ models, we present the corresponding expressions simultaneously and put label a for W_∞ model and label b for w_∞ model.

One dimensional W_∞ and w_∞ matrix model Lagrangians are given by (compare with (3.11) and (3.14) respectively):

$$\begin{aligned} L =& \mathrm{tr}\left[\frac{1}{2}(\partial_t \hat{M} - [\hat{\mathcal{A}}_0, \hat{M}])^2 - v(\hat{M})\right] \\ =& \frac{1}{2}\int d^2 z \big(\partial_t M(t,z,\bar{z}) - \{\!\{\mathcal{A}_0, M\}\!\}(t,z,\bar{z})\big) \mathrm{e}^{\partial_z \partial_{\bar{z}}} \big(\partial_t M(t,z,\bar{z}) - \{\!\{\mathcal{A}_0, M\}\!\}(t,z,\bar{z})\big) - \mathrm{tr} v(\hat{M}), \end{aligned} \tag{4.1a}$$

$$L=\int d\vec{\sigma}\left[\frac{1}{2}\big(\partial_t M(t,\vec{\sigma})-\epsilon^{ij}\partial_i\mathcal{A}_0(t,\vec{\sigma})\partial_j M(t,\vec{\sigma})\big)^2-v(M)\right]. \tag{4.1b}$$

The canonical quantization leads to the following Hamiltonians:

$$H=\int P(z,\bar{z})\partial_t M(z,\bar{z};t)d^2z-L=\int d^2z\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n!}\partial_z^n P(z,\bar{z})\partial_{\bar{z}}^n P(z,\bar{z})+\mathrm{tr}v(\hat{M}), \tag{4.2a}$$

$$H=\int P(\vec{x})\partial_t M(\vec{x};t)d^2x-L=\int d\vec{\sigma}\left(\frac{1}{2}P(\vec{\sigma})^2+v(M(\vec{\sigma}))\right), \tag{4.2b}$$

where P's are the canonical momentum operators conjugated to M's, with the following commutation relations:

$$[\hat{M}(z,\bar{z}),\hat{P}(z',\bar{z}')]=i\delta^{(2)}(z-z'), \tag{4.3a}$$

$$[\hat{M}(\vec{\sigma}),\hat{P}(\vec{\sigma}')]=i\delta(\vec{\sigma}-\vec{\sigma}'). \tag{4.3b}$$

The W_∞ (and w_∞) gauge invariance of the actions (4.1) leads to the following constraints, which we impose on the state vector $|\Psi\rangle$:

$$\hat{\Pi}_\xi|\Psi\rangle\equiv\int d^2z\{\!\{\xi,\hat{M}\}\!\}\hat{P}(z,\bar{z})|\Psi\rangle=0, \tag{4.4a}$$

$$\hat{\Pi}_\xi|\Psi\rangle\equiv\int d\vec{\sigma}\epsilon^{ij}\partial_i\xi(\vec{\sigma})\partial_j\hat{M}(\vec{\sigma})\hat{P}(\vec{\sigma})|\Psi\rangle=0, \tag{4.4b}$$

which simply state that the wave function (or the state vector) is gauge invariant and depends only on gauge invariant singlet variables. Therefore in order to solve the problem we choose the following W_∞ (and w_∞) invariant collective field as dynamical variables:

$$\phi(x)=\mathrm{tr}\delta(x-M(\hat{a},\hat{a}^\dagger)), \tag{4.5a}$$

$$\phi(x)=\int d\vec{\sigma}\delta\big(x-M(\vec{\sigma})\big). \tag{4.5b}$$

We then change variables from $P(z,\bar{z})$, $M(z,\bar{z})$ to $\pi(x)$, $\phi(x)$, where $\pi(x)$ is a canonical momentum conjugate to $\phi(x)$:

$$[\pi(x),\phi(x)]=-i\delta(x-x')+\mathrm{const.}. \tag{4.6}$$

The standard procedure (see A2) to do this is to compute $\Omega(x,x')$ and $\omega(x)$. We describe this calculation in Appendix A3. The definitions and the results are

$$\Omega(x,x') \equiv -\int d^2z \sum_{n=0}^{\infty} \frac{1}{n!}[\partial_{\bar{z}}^n P(z,\bar{z}),\phi(x)][\partial_z^n P(z,\bar{z}),\phi(x')] = \partial_x\partial_{x'}[\delta(x-x')\phi(x)], \tag{4.7a}$$

$$\Omega(x,x') \equiv -\int d\vec{\sigma}[P(\vec{\sigma}),\phi(x)][P(\vec{\sigma}),\phi(x')] = \partial_x\partial_{x'}\big[\delta(x-x')\phi(x)\big], \tag{4.7b}$$

and

$$\omega(x) \equiv \int d^2z \sum_{n=0}^{\infty} \frac{1}{n!}[\partial_{\bar{z}}^n P(z,\bar{z}),[\partial_z^n P(z,\bar{z}),\phi(x)]] = 2\partial_x[\phi(x)G(x;\phi)], \tag{4.8a}$$

$$\omega(x) \equiv \int d\vec{\sigma}[P(\vec{\sigma}),[P(\vec{\sigma}),\phi(x)]] = -\kappa^2\partial_x^2\phi(x), \tag{4.8b}$$

where $\kappa^2 = \delta^2(0)$, and $G(x;\phi) = P\int \frac{\phi(x')}{x-x'}$. Notice that we obtained the same expression for Ω for both theories but quite different expressions for ω.

In the collective field theory the hermiticity requirement of the Hamiltonian leads to the following equation for the Jacobian J of change of variables:

$$\omega(x) + 2\int dx'\Omega(x,x')C(x') = 0, \quad C(x) = \frac{1}{2}\frac{\delta}{\delta\phi(x)}J, \tag{4.9}$$

Using (4.7) and (4.8) and assuming $\partial_x\phi(-\infty) = \phi(-\infty)\partial_x C(-\infty) = 0$ we obtain

$$\partial_x C(x) = G(x;\phi), \tag{4.10a}$$

$$\partial_x C(x) = -\frac{1}{2}\kappa^2\frac{\partial_x\phi(x)}{\phi(x)} = -\frac{1}{2}\kappa^2\partial_x \ln\phi(x). \tag{4.10b}$$

Since the kinetic energy part of the hermitian Hamiltonian in the collective field theory is given by

$$K = \frac{1}{2}\int\int dxdx'\big[\pi(x)\Omega(x,x')\pi(x') + C(x)\Omega(x,x')C(x')\big], \tag{4.11}$$

using (4.7) and (4.10) we obtain the following Hamiltonians:

$$H = \int dx\left(\frac{1}{2}(\partial_x\pi(x))^2\phi(x) + \frac{\pi^2}{6}\phi(x)^3 + v(x)\phi(x)\right) - e\left(\int dx\phi(x) - N\right) \tag{4.12a}$$

$$H = \int dx \left(\frac{1}{2} \partial_x \pi(x) \phi(x) \partial_x \pi(x) + \frac{1}{8} \kappa^4 \frac{(\partial_x \phi(x))^2}{\phi(x)} + \tilde{v}(x)\phi(x) \right) - e \left(\int dx \phi(x) - L^2 \right) \tag{4.12b}$$

where e is a Lagrange multiplier to insure

$$\int dx\, \phi(x) = \mathrm{tr}1 \equiv N \qquad (\text{for } W_\infty), \tag{4.13a}$$

$$\int dx\, \phi(x) = \int d\vec{\sigma} \equiv L^2 \qquad (\text{for } \omega_\infty) \tag{4.13b}$$

which follows from the definition of collective field (4.5). Notice that eventually we have to take $N \to \infty,\ \kappa \to \infty,\ L \to \infty$. For this purpose we first make the following scale transformations:

$$x \to N^{1/2} x, \qquad \phi(x) \to N^{1/2}\phi(x), \quad \pi(x) \to N^{-1}\pi(x), \qquad e \to Ne \tag{4.14a}$$

$$x \to \kappa x, \qquad \phi(x) \to L^2 \kappa \phi(x), \quad \pi(x) \to L^{-2}\pi(x), \qquad e \to \kappa^2 e \tag{4.14b}$$

which preserve the canonical commutation relation (4.6). We obtain

$$H = \int dx \left[\frac{1}{2N^2} (\partial_x \pi(x))^2 \phi(x) + N^2 \left(\frac{\pi^2}{6} \phi(x)^3 + u(x)\phi(x) - e\phi(x) \right) \right] + N^2 e \tag{4.15a}$$

$$H = \int dx \left[\frac{1}{2N^2} (\partial_x \pi(x))^2 \phi(x) + N^2 \left(\frac{1}{8} \frac{(\partial_x \phi(x))^2}{\phi(x)} + \tilde{u}(x)\phi(x) - e\phi(x) \right) \right] + N^2 e \tag{4.15b}$$

where for w_∞ we set $N \equiv L\kappa$ and

$$u(x) = \sum_n N^{\left(\frac{n}{2}-1\right)} g_n x^n \equiv \sum_n \alpha_n x^n, \tag{4.16a}$$

$$\tilde{u}(x) = \sum_n \kappa^{n-2} \tilde{g}_n x^n = \sum_n (\kappa l)^{n-2} g_n x^n \equiv \sum_n \tilde{\alpha}_n x^n. \tag{4.16b}$$

We know from the previous study [13, 15] that the $1/N$ expansion of Hamiltonians (4.15) is a standard semi-classical expansion and in the $N \to \infty$ limit the excitation spectrum is finite provided that $u(x)$ is finite and given by

$$H = \frac{1}{2} \sum_{n=0}^{\infty} (p_n^2 + \omega_n^2 q_n^2), \qquad [q_n, p_m] = i\delta_{nm}, \tag{4.17}$$

where

$$\omega_n = n\pi/T, \quad T = \int_{\tilde{x}_1}^{\tilde{x}_2} \frac{dx'}{\sqrt{2(\tilde{e}_0 - u(x'))}}, \qquad \frac{1}{\pi}\int_{\tilde{x}_1}^{\tilde{x}_2} dx' \sqrt{2(\tilde{e}_0 - u(x'))} = 1, \tag{4.18a}$$

$$\omega_n = E_n - E_0, \qquad \left(-\frac{1}{2}\partial_x^2 + \tilde{u}(x)\right)\chi_n(x) = E_n \chi_n(x) \tag{4.18b}$$

The derivation of these results will be given in detail in Appendix A4 and A5.

The result for W_∞ model exactly coincides with the $N \to \infty$ limit of the matix model discussed in [13, 15]. Of cource it is expected already from the equations (4.7a) and (4.8a) since these equations coincide with the ones obtained in [13]. The infinite volume of color space (i.e. N) is absorbed by the multiplicative renormalization into the coupling constants (see (4.16)).

The W_∞ effective Hamiltonian (4.15a) is actually an element of the w_∞ spectrum generating algebra [1]. However this w_∞ symmetry is not directly related to the original W_∞ gauge symmetry of the Lagrangian, since the dynamical w_∞ transformations are realized non-trivially in the physical Hilbert space while the original W_∞ gauge symmetry acts trivially in the physical Hilbert space.

As we see in (4.18) the energy spectrum of W_∞ model is that of bosons with equally spaced frequencies irrespective of its interactions. For the w_∞ model the equally spaced spectrum occurs only when $\tilde{\alpha}_n = 0$ for $n \neq 2$, namely the free w_∞ theory. It appears that in quantum theory the $l \to \infty$ contraction of W_∞ theory is a free w_∞ theory. Therefore, it is not possible to learn W_∞ theory by studying w_∞ theory.

Acknowledgements

One of us (An.K.) would like to acknowledge useful discussions with V.P. Nair and G. Alexanian, and grateful to the High Energy group for constant interest in this work and financial support. The other (B. S.) would like to thank Y. Nagaoka and K. Shizuya for their hospitality extended to him at Yukawa Institute for Theoretical Physics of Kyoto University, where part of the work was done in the fall of 1995 under the sponsorship of

JSPS Fellowship. This work was supported by the National Science Foundation, grant number PHY - 9420615.

Appendix A1. Coherent state representation

We list the definitions and basic properties of the coherent state representation which we use throughout the paper:

$$
\begin{aligned}
&|z\rangle = e^{\hat{a}^\dagger z}|0\rangle, \qquad \langle z| = \langle 0|e^{\hat{a}\bar{z}}, \qquad \langle z'|z\rangle = e^{\bar{z}' z} \\
&\hat{a}|z\rangle = z|z\rangle, \qquad \langle z|\hat{a}^\dagger = \langle z|\bar{z}, \qquad \int d^2 z e^{-|z|^2}|z\rangle\langle z| = 1, \qquad d^2 z \equiv \frac{dRez\ dImz}{\pi}
\end{aligned}
\tag{A1.1}
$$

For a given real function $\xi(z,\bar{z}) = \sum_{m,n} \xi_{mn} z^n \bar{z}^m$ we obtain

$$
\xi(\hat{a},\hat{a}^\dagger) = \sum_{m,n} \xi_{mn}\hat{a}^n(\hat{a}^\dagger)^m = \int d^2 z e^{-|z|^2} \sum_{m,n} \xi_{mn}\hat{a}^n|z\rangle\langle z|(\hat{a}^\dagger)^m = \int d^2 z e^{-|z|^2}|z\rangle\xi(z,\bar{z})\langle z|
\tag{A1.2}
$$

A proof of (2.2) goes as follows.

$$
\begin{aligned}
\xi_1(\hat{a},\hat{a}^\dagger)\xi_2(\hat{a},\hat{a}^\dagger) &= \int d^2 z e^{-|z|^2} \int d^2 z' e^{-|z'|^2} |z\rangle\xi_1(z,\bar{z})\epsilon^{\bar{z}z'}\xi_2(z',\bar{z}')\langle z'| \\
&= \int d^2 z e^{-|z|^2} \int d^2 z' e^{-|z'|^2} |z\rangle\xi_1(z,\bar{z})\xi_2(-\overleftarrow{\partial}_{\bar{z}},\bar{z}')\epsilon^{\bar{z}z'}\langle z'| \\
&= \int d^2 z|z\rangle\xi_1(z,\bar{z})\xi_2(z-\overleftarrow{\partial}_{\bar{z}},\bar{z})\epsilon^{-|z|^2}\langle z| \\
&= \int d^2 z e^{-|z|^2}|z\rangle \sum_{n=1}^{\infty}\frac{(-)^n}{n!}\partial_{\bar{z}}^n\xi_1(z,\bar{z})\partial_z^n\xi_2(z,\bar{z})\langle z| \\
&= \ddagger \sum_{n=1}^{\infty}\frac{(-)^n}{n!}\partial_{\bar{z}}^n\xi_1(z,\bar{z})\partial_z^n\xi_2(z,\bar{z})\Big|_{\substack{z=\hat{a}\\ \bar{z}=\hat{a}^\dagger}}\ddagger.
\end{aligned}
\tag{2.2}
$$

A2. Change of variables in quantum mechanics [13]

We start with a standard form for the Hamiltonian and Schrödinger equation which is given by

$$
\hat{H}\psi \;=\; \Big(-\frac{1}{2}\sum_{a=1}^{N}\frac{\partial^2}{\partial q^{a\,2}} \;+\; V(\mathbf{q})\Big)\psi(q) \;=\; E\psi(q)
\tag{A2.1}
$$

We consider a transformation given by

$$
q^a \longrightarrow Q^a = f^a(q), \qquad q^a \;=\; F^a(Q)
\tag{A2.2}
$$

We use the chain rule of differentiation to convert the derivatives with respect to q's into derivatives with respect to Q's.

$$-\frac{1}{2}\sum_a \frac{\partial^2}{\partial q^{a\,2}}\psi(q) \;=\; \frac{1}{2}\Big(-\, i\sum_{ab}\frac{\partial^2 f^b}{\partial q^{a\,2}}\frac{\partial}{i\partial Q^b} \;+\; \sum_{abc}\frac{\partial f^b}{\partial q^a}\frac{\partial f^c}{\partial q^a}\frac{\partial}{i\partial Q^b}\frac{\partial}{i\partial Q^c}\Big)\psi(F(Q)) \tag{A2.3}$$

We define

$$\omega^a(Q) \equiv -\sum_b \frac{\partial^2 Q^a}{\partial q^{b\,2}} = -\sum_b \frac{\partial^2 f^a}{\partial q^{b\,2}}, \qquad \Omega^{ab}(Q) \equiv \sum_c \frac{\partial Q^a}{\partial q^c}\frac{\partial Q^b}{\partial q^c} = \sum_c \frac{\partial f^a}{\partial q^c}\frac{\partial f^b}{\partial q^c}. \tag{A2.4}$$

Then we obtain

$$\hat{H}\psi \;=\; \Big[\frac{i}{2}\big(i\sum_a \omega^a(Q)P_a \;+\; \sum_{ab}\Omega^{ab}(Q)P_aP_b\big) \;+\; \tilde{V}(Q)\Big]\psi(F(Q)), \tag{A2.5}$$

where we set $\frac{1}{i}\frac{\partial}{\partial Q^a} \;=\; P_a, \qquad \tilde{V}(Q) \;\equiv\; V(F(Q))$.

The Hamiltonian after the change of variables appears to be non-Hermitian if we take the naive Hermitian conjugate: ${P_a}^\dagger = P_a, {Q^a}^\dagger = Q^a$. This is because H is Hermitian in the original q-space. But after the change of variables, the Q-space should be defined by multiplying the wave function by the square root of the Jacobian in order to satisfy the naive Hermitian conjugation prescription:

$$\int dq\, \psi_1^*(q)\psi_2(q) \;=\; \int J(Q)\, dQ\, \psi_1^*(F(Q))\,\psi_2(F(Q)) = \int dQ\, \Psi_1^*(Q)\,\Psi_2(Q), \tag{A2.6}$$

where $\Psi(Q) = J^{1/2}(Q)\psi(F(Q))$. The Hamiltonian in Q-space is then obtained by a similarity transformation

$$H_{\rm eff} \;=\; J^{1/2} H J^{-1/2}, \tag{A2.7}$$

which should be Hermitian.

In practice the Jacobian is difficult to calculate while ω and Ω defined by (A2.4) are relatively easy to compute. So, it would be nice if $H_{\rm eff}$ is expressed in terms of ω and Ω. Notice first $J^\dagger(Q) = J(Q^\dagger) = J(Q)$, accordingly $J^{1/2}P_aJ^{-1/2} = P_a + iC_a(Q)$, where $C_a(Q) = \frac{1}{2}\frac{\partial}{\partial Q^a}\ln J(Q)$ and $C_a^\dagger = C_a$. We obtain

$$H_{\rm eff} = \frac{1}{2}\big[\, i\sum_a \omega^a(Q)(P_a \;+\; iC_a) + \sum_{ab}\Omega^{ab}(P_a \;+\; iC_a)(P_b \;+\; iC_b)\big] + \tilde{V}(Q). \tag{A2.8}$$

Since H_{eff} should be Hermitian, $H_{\text{eff}} - H_{\text{eff}}^{\dagger} = i\sum_a \bigl\{(\omega^a + 2\sum_b \Omega^{ab}C_b + \sum_b \Omega^{ab}_{\ ,b}), P_a\bigr\}_+ = 0$, and by taking a commutator bracket with Q_a we obtain

$$\omega^a \ + \ 2\sum_b \Omega^{ab}C_b \ + \ \sum_b \Omega^{ab}_{\ ,b} \ = \ 0. \tag{A2.9}$$

This is the equation that determines C_a. H_{eff} is then computed as

$$H_{\text{eff}} \ = \ \frac{1}{2}\sum_{ab}\bigl[P_a\Omega^{ab}P_b \ + \ C_a\Omega^{ab}C_b \ + \ (C_a\Omega^{ab})_{,b}\bigr] \ + \ \tilde{V} \tag{A2.10}$$

(A2.9) and (A2.10) are the main results.

A3. Calculation of Ω and ω

In order to calculate $\Omega(x,x')$ and $\omega(x)$ for W_∞ theory one needs the following equations:

$$\begin{aligned}
&[P(z,\bar{z}), e^{-ikM(\hat{a},\hat{a}^\dagger)}] = -e^{-|z|^2}k\int_0^1 d\tau e^{-ik\tau M(\hat{a},\hat{a}^\dagger)}|z\rangle\langle z|e^{-ik(1-\tau)M(\hat{a},\hat{a}^\dagger)},\\
&[P(z,\bar{z}), \phi(k)] = -ke^{-|z|^2}\langle z|e^{-ikM(\hat{a},\hat{a}^\dagger)}|z\rangle,\\
&[P(z,\bar{z}), [P(w,\bar{w}), \phi(k)]] = k^2 e^{-|z|^2-|w|^2}\int_0^1 d\tau\langle w|e^{-ik\tau M(\hat{a},\hat{a}^\dagger)}|z\rangle\langle z|e^{-ik(1-\tau)M(\hat{a},\hat{a}^\dagger)}|w\rangle.
\end{aligned} \tag{A3.1}$$

The proof is straightforward once we realize the following identity:

$$\epsilon^{-|z|^2}\sum_n \frac{1}{n!}\bigl((\partial_z - \bar{z})^n|z\rangle\bigr)\bigl((\partial_{\bar{z}} - z)^n\langle z|\bigr) = 1. \tag{A3.2}$$

The (A3.2) expresses the completeness property of the generators of W_∞ fundamental representation. It can be considered as a generalization of the completeness property of $SU(N)$ generators to the case of $N = \infty$. One can prove this identity by multiplying both sides by arbitrary ket vector $|z'\rangle$. Then the right hand side is $|z'\rangle$ and the left hand side is

equal to:

$$
\begin{aligned}
&\epsilon^{-|z|^2}\sum_n \frac{1}{n!}(\hat{a}^\dagger-\bar{z})^n|z\rangle(\partial_{\bar{z}}-z)^n\epsilon^{\bar{z}z'} \\
&=\sum_n \frac{1}{n!}\big((\hat{a}^\dagger-\bar{z})(z'-z)\big)^n|z\rangle\epsilon^{-|z|^2+\bar{z}z'} \\
&=\epsilon^{(\hat{a}^\dagger-\bar{z})(z'-z)}|z\rangle\epsilon^{-|z|^2+\bar{z}z'} \\
&=\epsilon^{\hat{a}^\dagger z'}\epsilon^{-\hat{a}^\dagger z}|z\rangle \\
&=|z'\rangle \qquad\qquad \text{(QED)}
\end{aligned}
\tag{A3.3}
$$

A4. Solution of W_∞ matrix model

Although we can solve this model starting from (4.15a) in a straightforward fashon [15] as we shall discuss it in A5 for w_∞ model, in this Appendix we solve it in a slightly different way which illuminates the dyanmical group structure of the theory.

We start with the Hamiltonian (4.12a) with the constraint (4.13a):

$$
H=\int dx\left(\frac{1}{2}(\partial_x\pi(x))^2\phi(x)+\frac{\pi^2}{6}\phi(x)^3+v(x)\phi(x)\right), \qquad \int dx\phi(x)=\mathrm{tr}1=N. \tag{A4.1}
$$

Using $y_\pm(x)=\pm\pi\phi(x)-\partial_x\pi(x)$ one [16] writes the Hamiltonian and the constraint as

$$
H=\frac{1}{2\pi}\int dx\int_{y_-(x)}^{y_+(x)}dy\left(\frac{1}{2}y^2+v(x)\right), \qquad \frac{1}{2\pi}\int dx\int_{y_-(x)}^{y_+(x)}dy=N, \tag{A4.2}
$$

where $y_\pm(x)$'s satisfy the commutation relation

$$
[y_\pm(x),y_\pm(x')]=\mp 2\pi i\delta'(x-x'). \tag{A4.3}
$$

We diagonalize this Hamiltonian by a canonical transformation. In the integral of (A4.2) we change variables from y,x to e,ξ such that $e=\frac{1}{2}y^2+v(x)$. The action integral is the generator of transformation $S(x,e)=\int^x ydy=\pm\int^x\sqrt{2(e-v(x'))}dx'$, accordingly $\xi=\frac{\partial S}{\partial e}=\pm\int^x\frac{dx'}{\sqrt{2(e-v(x'))}}$, where $\pm$ is for positive and negative ξ respectively. The boundary of the phase space is transformed from $y_\pm(x)$ to $e(\xi)$

$$
e(\xi)=\frac{1}{2}y_\pm^2(x)+v(x) \tag{A4.4}
$$

and the Hamiltonian and the constraint are given by

$$H = \frac{1}{2\pi}\oint d\xi \int^{e(\xi)} e de, \qquad \frac{1}{2\pi}\oint d\xi \int^{e(\xi)} de = N. \tag{A4.5}$$

We expand $e(\xi)$ around the minimum configuration e_0 of H: $e(\xi) = e_0 + \delta e(\xi)$, where e_0 is given by a solution of

$$\frac{1}{\pi}\int_{x_1}^{x_2} \sqrt{2(e_0 - v(x))} dx = N, \tag{A4.6}$$

where x_1 and x_2 are the turning points.

Since we see from (A4.6) that e_0 is of order N, we may assume $e_0 \gg \delta e(\xi)$ and we may approximate ξ on the boundary by $\xi(x) = \pm \int_{x_1}^{x} \frac{dx'}{\sqrt{2(e_0 - v(x'))}}$ and the half period by

$$T = \int_{x_1}^{x_2} \frac{dx'}{\sqrt{2(e_0 - v(x'))}} = \int_{\tilde{x}_1}^{\tilde{x}_2} \frac{dx'}{\sqrt{2(\tilde{e}_0 - u(x'))}}, \tag{A4.7}$$

where $\tilde{e}_0 = N^{-1} e_0, \quad \tilde{x}_1 = N^{-\frac{1}{2}} x_1$ and $u(x) = N^{-1} v(N^{\frac{1}{2}} x)$ (see (4.16a)).

Since the commutation rules of $e(\xi)$'s are given by

$$\begin{aligned}
[e(\xi), e(\xi')] &= \left[\frac{1}{2} y_{\pm}^2(x) + v(x), \frac{1}{2} y_{\pm}^2(x') + v(x')\right] \\
&= \mp 2\pi i y_{\pm}(x) y_{\pm}(x') \delta'(x - x') \\
&\approx -2\pi i \delta'(\xi - \xi'),
\end{aligned}$$

if we define θ and $r(\theta)$ by $\theta = \omega\xi, \quad r(\theta) = \omega^{-1} \delta e(\xi), \quad \omega = \frac{\pi}{T}$, we obtain

$$[r(\theta), r(\theta')] = -2\pi i \delta'(\theta - \theta') \tag{A4.8}$$

The normal mode expansion of $r(\theta)$ is given by $r(\theta) = \sqrt{\frac{2}{\omega}} \sum_{n>0} (\sin(n\theta) p_n + n\omega \cos(n\theta) q_n)$, and leads to the following Hamiltonian:

$$\begin{aligned}
&H = E_0 + H_{\text{coll}}, \\
&H_{\text{coll}} = \oint \frac{d\xi}{2\pi} \int_0^{\delta e(\xi)} e de = \omega \oint \frac{d\theta}{2\pi} \frac{1}{2} (r(\theta))^2 = \frac{1}{2} \sum_n (p_n^2 + \omega_n^2 q_n^2)), \\
&\omega_n = n\omega = n\frac{\pi}{T}.
\end{aligned} \tag{A4.9}$$

From (A4.6) and (A4.7) it is obvious that T^{-1} is finite in the large N limit provided $u(x)$ is finite. In the double scaling limit one of the turning points goes to infinity so that $T \to \infty$ and we obtain the continuous spectrum (chiral Boson).

The reason why we could solve the W_∞ model by a canonical transformation is that there exists a dynamical w_∞ algebra in the physical Hilbert space and the Hamiltonian is a generator of the algebra. We simply quote a result of [2], namely $\rho[\xi]$'s defined by

$$\rho[\xi] = \oint \frac{d\theta}{2\pi} \int^{r(\theta)} dr \xi(r,\theta), \tag{A4.10}$$

satisfy the w_∞ commutation relation (2.6).

A5. w_∞ Matrix Model

We first obtain the field configuration at which the potential energy of (4.15b) is minimum. The equations are $-\frac{1}{4}\partial\left(\frac{\partial\phi}{\phi}\right) - \frac{1}{8}\left(\frac{\partial\phi}{\phi}\right)^2 + \tilde{u}(x) = e$ and $\int dx \phi(x) = 1$. For the variable $\varphi(x) = \sqrt{\phi(x)}$ the first equation is the Scrödinger equation $\left(-\frac{1}{2}\partial^2 + \tilde{u}(x)\right)\varphi(x) = e\varphi(x)$ and the second equation is the normalization of the wave function. Therefore we consider an orthonormal set of eigenfunction χ_n with eigenvalue E_n. We set $\phi(x) = \chi_0^2(x) + \frac{1}{\sqrt{N}}\eta(x), \quad \pi(x) = \sqrt{N}\zeta(x)$ and expand the Hamiltonian. We obtain

$$H = \frac{1}{2}\int dx \left[\chi_0^2(x)\big(\partial\zeta(x)\big)^2 + \frac{1}{4}\left(\frac{(\partial\eta(x))^2}{\chi_0^2(x)} + (2\partial^2 \ln \chi_0)\frac{\eta^2(x)}{\chi_0^2(x)}\right)\right] \tag{A5.1}$$

$\eta(x)$ and $\zeta(x)$ satisfy the canonical commutation relation:

$$[\partial\zeta(x), \eta(x')] = -i\delta'(x-x') \tag{A5.2}$$

and the constraint $\int dx \eta(x) = 0$. The normal mode expansion is now straightforward:

$$\eta(x) = \chi_0(x)\sum_{n=1}^{\infty} \sqrt{2\omega_n}\chi_n(x)\, q_n, \qquad \zeta(x) = \chi_0^{-1}(x)\sum_{n=1}^{\infty}\frac{1}{\sqrt{2\omega_n}}\chi_n(x)\, p_n. \tag{A5.3}$$

and

$$H = \frac{1}{2}\sum_{n=0}^{\infty}\big(p_n^2 + \omega_n^2 q_n^2\big), \qquad [q_n, p_m] = i\delta_{n,m}\ , \qquad \omega_n = E_n - E_0. \tag{A5.4}$$

References

[1] J. Avan and A. Jevicki, *Phys. Lett.* **B266** (1991) 35; *Phys. Lett.* **B272** (1991) 17;

A. Gerasimov, A. Marshakov, A. Mironov, A. Morozov and A. Orlov, *Nucl. Phys.* **B357** (1991) 565;

D. Minic, J. Polchinsky and Z. Yang, *Nucl. Phys.* **B362** (1991) 125;

G. Moore and N. Seiberg, *Int. J. Mod. Phys.* **A8** (1992) 2601;

I. Klebanov and A.M. Polyakov, *Mod. Phys. Lett.* **A6** (1991) 3273;

E. Witten, *Nucl. Phys.* **B373** (1992) 187;

S.R. Das, A. Dhar, G. Mandal and S.R. Wadia, Int. J. Mod. Phys. **A7** (1992) 5165;

[2] S. Iso, D. Karabali and B. Sakita , *Phys. Lett.* **B296** (1992) 143

[3] A. Cappelli, C. Trugenberger and G. Zemba, *Nucl. Phys.* **B396** (1993) 465

[4] S.R. Das, A. Dhar, G. Mandal and S.R. Wadia, *Int. J. Mod. Phys.* **A7** (1992) 5165;

A. Dhar, G. Mandal and S.R. Wadia, *Mod. Phys. Lett.* **A7** (1992) 937;

[5] B. Sakita , *Phys. Lett.* **B315** (1993) 124

K. Shizuya, *Phys. Rev.* **B52** (1995) 2747

[6] E. G. Floratos and J. Iliopoulos, *Phys. Lett.* **B201** (1988) 237;

E. G. Floratos, *Phys. Lett.* **B228** (1989) 335

[7] E. G. Floratos, J. Iliopoulos and G. Tiktopoulos, *Phys. Lett.* **B217** (1989) 285;

[8] D. B. Fairlie, P. Fletcher and C. K. Zachos, *Phys. Lett.* **B218** (1989) 203;

D. B. Fairlie and C. K. Zachos, *Phys. Lett.* **B224** (1989) 101;

D. B. Fairlie, P. Fletcher and C. K. Zachos, *J. Math. Phys.* **31** (1990) 1088;

C. K. Zachos *Hamiltonian Flows, $SU(\infty), SO(\infty), USp(\infty)$, and Strings* in *Differential Geometric Methods inTheoretical Physics; Physics and Geometry*, NATO ASI Series, L.-L. Chau and W. Nahm (eds.), Plenum, New York, p.423, 1990

[9] G. 't Hooft, *Nucl. Phys.* **B72** (1974) 461; *Nucl. Phys.* **B75** (1974) 461

[10] See [2] [3]. For earlier work see

J. Hoppe, MIT Ph. D. Thesis (1982);

J. Hoppe and P. Schaller, *Phys. Lett.***B237** (1990) 407;

C.N. Pope, L.J. Romans and X. Shen *"A Brief History of W_∞"* in Strings 90, ed. R. Arnowitt et al (World Scientific 1991) and references therein.

[11] J. E. Moyal, *Proc. Camb. Phyl. Soc.* **45** (1949) 99

[12] I. Bars, *Phys. Lett.* **B245** (1990) 35

[13] A. Jevicki and B. Sakita, *Nucl. Phys.* **B165** (1980) 511. See also

B. Sakita, "*Quantum Theory of Many-Variable Systems and Fields*", World Scientific, Singapore (1985)

[14] S. J. Rankin, *Ann. Phys.* **218** (1992) 14

[15] M. Mondello and E. Onofri, *Phys. Lett.* **98B** (1981) 277

R. Jackiw and A. Strominger, *Phys. Lett.* **99B** (1981) 133

J. Shapiro, *Nucl. Phys.* **B184** (1981) 218

[16] J. Polchinski, *Nucl. Phys.* **B362** (1991) 125